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A quasi-exactly solvable Lipkin–Meshkov–Glick model

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Abstract
We prove that a special Lipkin–Meshkov–Glick model is quasi-exactly solvable with solutions that can be expressed in the $SU(2)$ coherent state form. Ground-state properties of the model are studied analytically. We also show that the model reduces to the standard two-site Bose–Hubbard model in the large-N limit for finite U/t or large $(N-1)|U|/t$ cases with finite N, which proves that in these cases the ground state of the standard two-site Bose–Hubbard model is an $SU(2)$ coherent state.

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1. Introduction

It is well known that the Lipkin–Meshkov–Glick (LMG) model [1–3] that originated from nuclear many-body problems is related to a broad range of other fields, such as spin systems [4], Bose–Einstein condensates [5], quantum entanglement [6], etc. The ground-state entanglement entropy has been analyzed [7]. Further, the large-N behavior of the model has been explored using a semiclassical approach [8]. The finite-size scaling behavior has also been investigated [9]. Though the model is known to be exactly solvable [10, 11], and the numerical analysis for systems with small number of bosons is possible, large-N cases have not been reachable with the techniques used up until now. A special case of the model is also related to the standard two-site Bose–Hubbard model [12], which has been investigated widely by many authors using various methods, such as the Gross–Pitaevskii approximation [13], mean-field theory [14, 15], the quantum phase model [16] and the Bethe ansatz method [17]. The temporal evolution of the expectation value for the relative number of particles between the two condensates for different choices of the coupling parameter and distinct initial states has been analyzed [18]. Quantum phase transitions of the model have been studied by using the Bogoliubov approximation [19] and a Bethe ansatz solution for relatively low number of system bosons [20]. Chaotic and semi-classical properties of a similar model have also been presented [21, 22].

Generally speaking, quantum phase transitions in a quantum many-body system should be defined in the thermodynamic (large-N) limit, although the transitional behavior may also be observable mesoscopically. Since large-N limit is not reachable, a coherent state algorithm, such as the one introduced by Gilmore based on an algebraic coherent state constructed according to the dynamical symmetry group of a system [23, 24], has often been adopted to investigate quantum phase transition phenomena, for example, in the interacting boson model for nuclei, which allows one to take the $N \to \infty$ limit called the classical or mean-field limit. It is commonly accepted that the algorithm used to analyze quantum phase transitions is reliable.

The purpose of this paper is twofold. First, we will show that a special Lipkin–Meshkov–Glick (LMG) model corresponding to an extended two-site Bose–Hubbard model is quasi-exactly solvable with solutions that can be expressed in terms of the $SU(2)$ coherent state form. Second, we will show that the ground state of the standard two-site Bose–Hubbard model in the large-N limit is indeed an $SU(2)$ coherent state under the condition $(N-1)|U|/t \gg 1$, which corresponds to a Mott-insulate phase. It follows from this that the $SU(2)$ coherent state cannot be used to describe the standard two-site Bose–Hubbard model at the critical point.

2. A quasi-exactly solvable case

We consider a special LMG model with the Hamiltonian

$$\hat{H} = -t(J_+ + J_-) + UJ_0^2 + \delta\left(J_+^2 + J_-^2\right), \tag{1}$$

where J_μ $(\mu = +, -, 0)$ are generators of the $SU(2)$ algebra, which satisfy commutation relations $[J_0, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_0$. Note that the Hamiltonian (1) corresponds to the extended two-site Bose–Hubbard model with the Jordan–Schwinger realization of the $SU(2)$ algebra: $J_+ = a^\dagger b$, $J_- = b^\dagger a$, $J_0 = (a^\dagger a - b^\dagger b)/2$, where $a^\dagger$ (a) and $b^\dagger$ (b) are boson creation (annihilation) operators in two traps or different hyperfine states. In this case, the first term in (1) describes the tunneling between the two sites, characterized by the strength parameter t; the second term in (1) is equivalent to an on-site single-particle energy which originates from the external potentials and is related to the on-site part of the kinetic energy, while the third term in (1) describes double-boson hopping between the two sites, characterized by the strength parameter δ. And indeed, as shown in [25] through a systematic expansion in powers of the lattice attenuation factor $\epsilon \ll 1$ for the Hamiltonian including one- and two-body interactions describing systems of interacting, ultra-cold, spin-zero, neutral bosonic atoms that are harmonically trapped and subject to an optical lattice potential, the final extended Bose–Hubbard Hamiltonian in the two-site case is equivalent to the form shown in (1) since the total number of bosons $\hat{N} = a^\dagger a + b^\dagger b$ is a conserved quantity. As shown in [25], $U/\delta \sim \epsilon^2$ is a small quantity and is often neglected. If there is no dipole interaction among the bosons, the parameter $\delta > 0(<0)$ when $U > 0(<0)$; however, the sign of δ may be the same as or opposite to that of U when dipole interaction among bosons must be considered [26].

Let $|JM\rangle$ be basis vectors of the $SU(2)$ irrep with $J = N/2$, where N is the total number of particles in the system. Generally, the Hamiltonian (1) can be diagonalized numerically in the Hilbert subspace spanned by $\{|JM\rangle\}$, which is $N + 1$ dimensional. However, it can be shown that

$$|z\rangle = e^{zJ_+}|J\ -J\rangle \tag{2}$$

is an eigenstate of (1) when the parameters δ and z satisfy a special condition. With the help of the following relations:

$$J_0 e^{zJ_+}|J\,-J\rangle = e^{zJ_+}(-J + zJ_+)|J\,-J\rangle, \tag{3a}$$

$$J_- e^{zJ_+}|J\,-J\rangle = e^{zJ_+}(2zJ - z^2 J_+)|J\,-J\rangle, \tag{3b}$$

$$J_0^2 e^{zJ_+}|J\,-J\rangle = e^{zJ_+}(J^2 - z(2J-1)J_+ + z^2 J_+^2)|J\,-J\rangle, \tag{3c}$$

$$J_-^2 e^{zJ_+}|J\,-J\rangle = e^{zJ_+}(2J(2J-1)z^2 - 2(2J-1)z^3 J_+ + z^4 J_+^2)|J\,-J\rangle, \tag{3d}$$

one can easily check that the eigen-equation $\hat{H}|z\rangle = E(z)|z\rangle$ requires that coefficients proportional to J_+ and J_+^2 are zero. The eigen-energy in this case is

$$E(z)/t = -2zJ + VJ^2 + 2\bar{\delta}z^2 J(2J-1), \tag{4}$$

where $V = U/t$. The zero-value condition for the coefficient proportional to J_+^2 gives

$$\bar{\delta} = \delta/t = -\frac{Vz^2}{1+z^4}, \tag{5}$$

which shows that the sign of $\bar{\delta}$ must be opposite to that of V, which corresponds to a situation when dipole interaction among bosons is taken into account [26], while the zero-value condition for the coefficient proportional to J_+ provides the ansatz equation to determine z with

$$z^6 + xz^5 - z^4 + z^2 - xz - 1 = 0, \tag{6}$$

where $x = V(2J-1)$, which provides for four real roots and two complex roots. Only four real roots correspond to possible solutions and the condition of the quasi-solvability, which are

$$z_{\mathrm{c},\pm} = \pm 1, \tag{7}$$

$$z_\pm = -\frac{x}{4} - \frac{1}{4}\sqrt{8+x^2} \pm \frac{1}{2}\sqrt{\frac{x^2}{2} - 2 + \frac{x}{2}\sqrt{8+x^2}} \tag{8a}$$

for $x \geqslant 1$ and

$$z_\pm = -\frac{x}{4} + \frac{1}{4}\sqrt{8+x^2} \pm \frac{1}{2}\sqrt{\frac{x^2}{2} - 2 - \frac{x}{2}\sqrt{8+x^2}} \tag{8b}$$

for $x \leqslant -1$.

According to relation (5), the first two solutions given by (7) lead to large amplitude of double-boson hopping with $\bar{\delta} = -V/2$, which may be irrelevant to real situations. The other two solutions, (8*a*) and (8*b*), correspond to weak double-boson hopping cases when the dipole interaction among bosons is considered as shown in [26].

It can be verified that $z_\pm < 0$ and $E(z) > 0$ is always satisfied for $V > 0$ with $x > 1$. Therefore, the solution (8*a*) corresponds to an excited state. Actually, it can be verified by direct diagonalization that $E(z)$ with $x \geqslant 1$ is the highest excited state in the system. When $V < 0$ with $x < -1$, $z_\pm > 0$ and $E(z_\pm) < 0$ is always satisfied. Though a direct proof is in demand, one can check numerically in comparison with direct diagonalization for any accessible J that $E(z_\pm)$ with $x \leqslant -1$ corresponds to the ground-state energy of the system. It should be noted that two solutions of (8*b*) denoted as z_- and z_+ with $0 < z_- < z_+$ and $z_- z_+ = 1$ result from the S_2 symmetry with respect to the site exchange $a \rightleftharpoons b$ in the system when $U < 0$, namely $|z_-, (a,b)\rangle = |z_+, (b,a)\rangle$, which provides for the same ground state of the system with $E(z_+) = E(z_-)$. Therefore, in the following, we set $z = z_+$ when $x \leqslant -1$ and only focus on the case where the on-site interaction is attractive with $U < 0$ similar to the case studied in [20], in which the effective interaction energy for the internal Josephson dynamics is negative when the model is used to describe two superconductors connected with

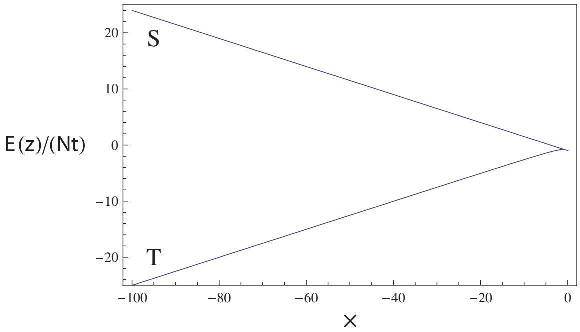

Figure 1. The ground-state energy per particle as a function of x, where S denotes the super-fluid region, while T denotes the transitional region. The crossing point of S and T shows the critical point in the system with $E^{(c)}(z)/(Nt) = -3/4$ at $x = -1$.

a Josephson junction [27]. With this choice, the ground-state properties of the model described by the Hamiltonian (4) can be studied by using the coherent state ansatz (2).

It can be shown from (8b) that $z = z_{c,+} = 1$ with $\delta = -U/2$ is also a ground-state solution with $\delta = -U/2 = 0$. Therefore, $z = z_{c,+} = 1$ with $\delta = -U/2$ corresponds to the super-fluid phase of the system, which is independent of x, and is only characterized by the condition $\delta = -U/2$. The transitional phase is characterized by $-\infty < x = (N-1)U/t < -1$, and the Mott-insulate phase is characterized by $x = (N-1)U/t \to -\infty$. Hence, $x_c = (N-1)V_c = -1$ corresponding to $z = z_{c,+} = 1$ is the critical point of the system in this case.

One can easily verify that the ground-state energy per particle in both the superfluid region with $\delta = -U/2$ and the transitional region with $-\infty < x < -1$ is linear in x in the large-N limit. The ground-state energy per particle in the superfluid region is $E^{(S)}(z)/(Nt) \approx -1-x/4$ with $-\infty < x < 0$ in the large N limit, while it is $E^{(T)}(z)/(Nt) \approx 3x/4$ in the transitional region with $-\infty < x \leqslant -1$, which varies as a function of x in the two different phases as shown in figure 1. Though the ground-state energy per particle in both regions are the same at the critical point, it is obvious that the first derivative of the ground-state energy per particle in the super-fluid region with respect to x and that in the transitional regions are different at the critical point $x_c = -1$, which shows that the super-fluid to the Mott-insulate phase transition is a first-order quantum phase transition. The first derivatives of other quantities, such as the occupation probabilities of a- and b-bosons, local particle number fluctuations, etc, with respect to the control parameter x in both regions, are also different at the critical point $x = x_c$.

Since the ground state can be expressed as an $SU(2)$ coherent state, many ground-state properties of the model can be studied analytically by using the following results:

$$\langle z'|z\rangle = (1 + z'z)^N, \tag{9a}$$

$$\langle z'|J_0|z\rangle = \left(-J + z\frac{\partial}{\partial z}\right)\langle z'|z\rangle, \tag{9b}$$

$$\langle z'|J_+|z\rangle = \frac{\partial}{\partial z}\langle z'|z\rangle, \tag{9c}$$

$$\langle z'|J_0^2|z\rangle = \left(J^2 - z(2J-1)\frac{\partial}{\partial z} + z^2\frac{\partial^2}{\partial z^2}\right)\langle z'|z\rangle, \tag{9d}$$

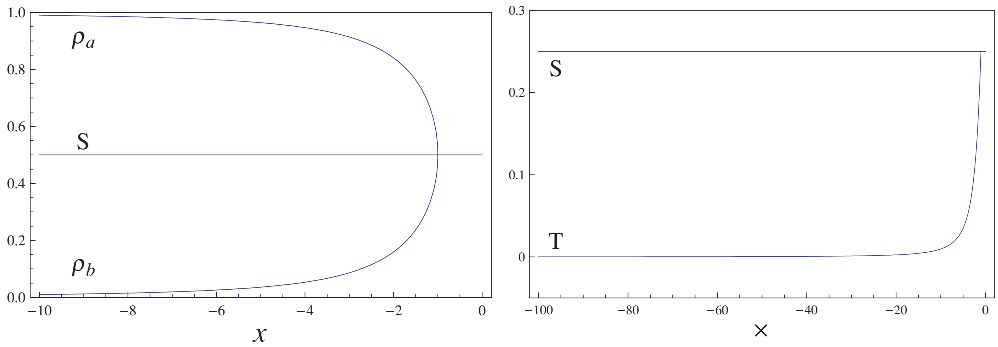

Figure 2. The occupation probabilities of a- and b-bosons (10) and local particle number fluctuations (11) as functions of $x = (N-1)U/t$. The left panel shows the occupation probabilities of a- and b-bosons, ρ_a and ρ_b in the transitional region, and those in the super-fluid region denoted as S with $\rho_a^{(S)} = \rho_b^{(S)} = 0.5$. The right panel shows the local particle number fluctuation $\overline{(\Delta n_a)^2}/N = \overline{(\Delta n_b)^2}/N$ in the super-fluid (S) and transitional (T) regions.

according to the relations shown in (3). For example, the occupation probabilities of a- and b-bosons are given by

$$\rho_a = \frac{1}{N}\langle z|a^\dagger a|z\rangle/\langle z|z\rangle = \frac{1}{N}\langle z|J_0|z\rangle/\langle z|z\rangle + \frac{1}{2} = \frac{z^2}{1+z^2}, \tag{10a}$$

$$\rho_b = \frac{1}{N}\langle z|b^\dagger b|z\rangle/\langle z|z\rangle = \frac{1}{2} - \frac{1}{N}\langle z|J_0|z\rangle/\langle z|z\rangle = \frac{1}{1+z^2}, \tag{10b}$$

and the local particle number fluctuation at site a or b is the same with

$$\overline{(\Delta n_a)^2} = \overline{(\Delta n_b)^2} = \frac{Nz^2}{(1+z^2)^2}, \tag{11}$$

where $\Delta n_a = a^\dagger a - N\rho_a$, $\Delta n_b = b^\dagger b - N\rho_b$ and $\overline{A} = \langle z|A|z\rangle/\langle z|z\rangle$ for any operator A. Figure 2 shows the occupation probabilities of a- and b-bosons as functions of $x = (N-1)U/t$ with $z = z_+$ in the transitional region and $z = 1$ in the super-fluid region, from which it is shown that the occupation probabilities of a- and b-bosons are the same with $\rho_a = \rho_b = 0.5$ in the super-fluid region with $\delta = -U/2$, while more and more particles will occupy one of the sites with increasing values of $(N-1)|U|/t$ toward the Mott-insulate phase. It becomes a Mott-insulate when $x = (N-1)U/t \to -\infty$. The local particle number fluctuation is proportional to the total number of particles in the system with $\overline{(\Delta n_a)^2} = \overline{(\Delta n_b)^2} = N/4$ in the super-fluid region, while it drastically decreases with increasing values of $(N-1)|U|/t$ in the transitional region, and becomes negligible when $x \ll -1$.

As is known, the entanglement measure for any pure bipartite system is defined by

$$\eta = -\mathrm{Tr}(\rho_a \log_{N+1} \rho_a) = -\mathrm{Tr}(\rho_b \log_{N+1} \rho_b), \tag{12}$$

where ρ_a is the reduced density matrix obtained by taking the partial trace over the subsystem b, and similarly for ρ_b. We use the logarithm to the base $N+1$ instead of base 2 to ensure that the maximal measure is normalized to 1. The explicit form of (12) for the present case is

$$\eta = -\sum_{k=0}^{N} \binom{N}{k} \frac{z^{2k}}{(1+z^2)^N} \log_{N+1}\left(\binom{N}{k} \frac{z^{2k}}{(1+z^2)^N}\right), \tag{13}$$

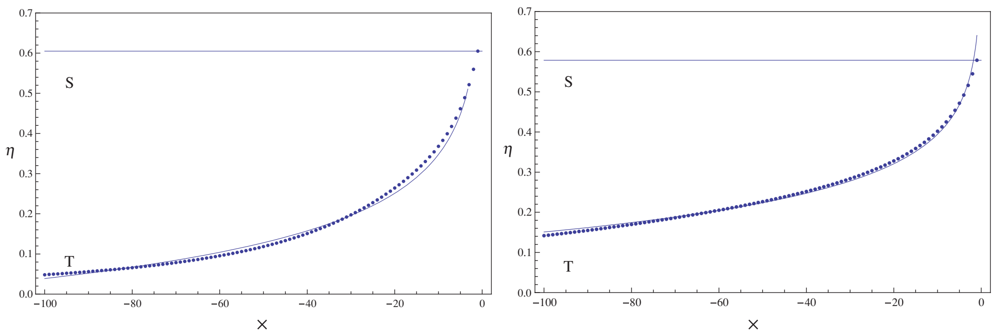

Figure 3. The entanglement measure (13) of the model as a function of x. The left panel shows the measure in the super-fluid (*S*) (straight solid line) and transitional (*T*) (dotted curve) regions with $N = 1000$ particles. In the transitional region, the solid curve indicates the empirical fit to (13) with $\eta(x) = -2.491\,78 + 3.185\,86/|x|^{0.05}$. The right panel shows the measure with $N = 10\,000$ particles, in which the solid curve in the transitional region indicates the empirical fit to (13) with $\eta(x) = -20.8643 + 21.5047/|x|^{0.005}$.

Table 1. The ground-state energy per particle $E(z)/(Nt)$ of the extended Bose–Hubbard model (EBHM) with the Hamiltonian shown in (1) and the standard Bose–Hubbard model (BHM) with $\delta = 0$ for $U/t = -0.011$ and different number of particles.

	BHM	EBHM		BHM	EBHM
$N = 100$	$-1.003\,28$	$-0.760\,457$	$N = 1000$	-2.841	$-2.839\,54$
$N = 300$	$-1.129\,16$	$-1.087\,23$	$N = 3000$	$-8.280\,31$	$-8.280\,26$
$N = 500$	-1.5572	$-1.546\,47$	$N = 5000$	-13.7682	-13.7682

which depends not only on x but also on N. It is obvious that η decreases with increasing values of $|x|$ or N. A simple fit to (13) for different large-N cases shows that $\eta \approx c_1 + c_2/|x|^{\alpha}$, where c_1, c_2 and $\alpha > 0$ are coefficients which are N-dependent. Figure 3 shows the entanglement measure (13) of the system as a function of x and an empirical fit to the curve for $N = 1000$ and 10 000, respectively.

Furthermore, the Hamiltonian (1) will become the standard Bose–Hubbard model in the large-N limit for finite U/t or large $(N-1)|U|/t$ cases with finite N, namely the third term in (1) will be negligible in such cases. Hence, $|z_{\pm}\rangle$ tends to be the ground state of the standard Bose–Hubbard model in the large-N limit for finite U/t or large $(N-1)|U|/t$ cases with finite N. To prove this, we calculate its contribution to the ground-state energy, which is

$$\delta\langle z_+|(J_+^2 + J_-^2)|z_+\rangle = -2Nx\, \left.\frac{z_+^4}{(1+z_+^4)(1+z_+^2)^2}\right|_{x\ll -1} \to -2N/x^3. \tag{14}$$

The result indicates that the ground-state energy of the extended Bose–Hubbard model only deviates from that of the standard Bose–Hubbard model by a term of $\mathrm{O}((U/t)^{-3}N^{-2})$ for large-N cases. Therefore, the third term of (1) is indeed negligible in the large-N limit for finite U/t or large $(N-1)|U|/t$ cases with finite N. Table 1 shows the ground-state energies per particle of the extended Bose–Hubbard model with the Hamiltonian shown in (1) and that of the standard Bose–Hubbard model with $\delta = 0$ for $U/t = -0.011$ fixed and $N = 100$, 300, 500, 1000, 3000 and 5000, respectively, which indicates that the ground-state energies of the two models become the same when $N \geqslant 5000$ in this case. Table 2 shows the ground-state energies per particle of the two models for $N = 500$, 1000 and 3000, respectively, with

Table 2. Ground-state energy per particle $E(z)/(Nt)$ of the extended Bose–Hubbard model (EBHM) with the Hamiltonian shown in (1) and the standard Bose–Hubbard model (BHM) with $\delta = 0$ for $N = 500$, 1000 and 3000 and different values of $x = (N-1)U/t$ with $U < 0$.

	$N = 500$		$N = 1000$		$N = 3000$	
	BHM	EBHM	BHM	EBHM	BHM	EBHM
$x = -5$	−1.452 52	−1.438 65	−1.451 26	−1.437 39	−1.450 42	−1.436 56
$x = -10$	−2.720 91	−2.603 09	−2.6025	−2.600 58	−2.600 83	−2.598 91
$x = -20$	−5.060 25	−5.059 77	−5.055	−5.054 76	−5.051 67	−5.051 42
$x = -30$	−7.548 36	−7.548 29	−7.540 83	−7.540 77	−7.535 83	−7.533 76
$x = -40$	−10.045	−10.045	−10.035	−10.035	−10.0283	−10.0283

different values of x, which indicates that the ground-state energies of the two models become the same when $x \leqslant -40$ for these cases. In these two tables, the ground-state energy of the standard Bose–Hubbard model is obtained by direct matrix diagonalization.

3. Concluding remarks

In summary, we have shown that a special Lipkin–Meshkov–Glick (LMG) model that is equivalent to the extended two-site Bose–Hubbard model is quasi-exactly solvable. The quasi-exact solvability requires that the double-boson hopping strength δ and the on-site interaction parameter U satisfy a restricted condition. When the on-site interaction is attractive, the ground state of the model is an $SU(2)$ coherent state. The ground-state properties of the model, such as ground-state energy, local particle occupation probabilities and particle number fluctuations, and entanglement measures of the system are studied analytically, which show that the model has three distinct phases, denoted as the super-fluid, transitional and Mott-insulate. With increasing values of $(N-1)|U|/t$, the system undergoes a first-order quantum phase transition with the critical point $|U_c|/t = 1/(N-1)$, and becomes the Mott-insulate when $(N-1)|U|/t \to \infty$. Furthermore, our analysis shows that the model becomes the standard two-site Bose–Hubbard model in the large-N limit for finite U/t or large $(N-1)|U|/t$ cases with finite N, which proves that the ground state of the standard two-site Bose–Hubbard model becomes an $SU(2)$ coherent state in such cases.

More importantly, it is now clear that an $SU(2)$ coherent state is the ground state of the standard two-site Bose–Hubbard model in the large-N limit only when $(N-1)|U|/t \gg 1$ corresponding to a Mott-insulate phase. As has been shown in [20], the critical point of the standard Bose–Hubbard model is given by $(N-1)|U|/t \sim 1$ based on the results obtained with finite N, of which the ground state cannot be described by an $SU(2)$ coherent state. $SU(2)$ is just the dynamical symmetry group of the standard two-site Bose–Hubbard model. Therefore, one should be cautious when the coherent state algorithm [23, 24] or other similar semi-classical methods are applied to study critical point phenomena in quantum phase transitions of many-body systems. Analysis of other integrable models, such as the one related to atom-molecule Bose–Einstein condensate context [28], may be also possible. Related work is in progress.

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